An Improvement of Convergence in Finite element Analysis with Infinite Element Using Deflation

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Abstract — In finite element analyses for electromagnetic field, large air regions should be divided by FE meshes. It causes to increase analysis time. The infinite element which can express the electromagnetic field in infinite region is proposed in order to solve this problem. However, by using this element, the convergence characteristic of the Incomplete Cholesky Conjugate Gradient (ICCG) method deteriorates because the condition number of system matrix becomes large. Thus, in this paper, we introduce a deflation technique to improve the convergence characteristic. Numerical examples show that the deflation technique can improve convergence characteristic of a magnetostatic analysis with finite and infinite elements.

I. INTRODUCTION

In the numerical analyses of electromagnetic field, the finite element (FE) method is widely used. The electromagnetic phenomena spread over infinite space. Therefore, this method needs to divide the air region by FE meshes. This causes to increase analysis time and cost of mesh generation. One of the solutions of this problem is the infinite element (IE) technique [1],[2]. This technique enables to reduce the mesh in air region. However, the IE technique has a disadvantage that the condition number of system matrix becomes large and the convergence characteristic of ICCG method deteriorates. This results in increasing the computational time. For this reason, improving convergence characteristic of ICCG is required.

One of the techniques to improve the convergence characteristic of ICCG is the deflation technique [3],[4]. Small eigenvalues of the system matrix are replaced with zeros to improve the condition number. In this paper, the deflation technique is applied to FE analysis using IE technique.

II. INFINITE ELEMENT TECHNIQUE

Let's us consider a IE proposed in Ref. [1],[2]. The IE is based on an idea that interpolation of radiation direction is expressed in a multipole expansion of potential. The IE is connected with boundary of FE region: the surface of FE corresponds to the IE element. The matrix obtained by the weak form of Galerkin method using IE is sparse and symmetric. Thus, the system matrix made from IE can be solved by the ICCG method.

Figure 1 shows the local coordination of a square IE for hexahedral FE.

The IE consists of 4 edges on the square and 4 edges whose end are on the vertexes of square. The latter edges extend to infinity. The reference point O (X_0, Y_0, Z_0) is placed in the FE region. The local coordinate system consists of r, s and t; $-1 < r, s < 1$, $1 < t < \infty$.

The coordinate of an arbitrary point P in IE is given by:

$$\begin{aligned} x &= X_0 + \sum_i \frac{1}{4}(1+r_i r)(1+s_i s)t(x_i - X_0) \\ y &= Y_0 + \sum_i \frac{1}{4}(1+r_i r)(1+s_i s)t(y_i - Y_0) \\ z &= Z_0 + \sum_i \frac{1}{4}(1+r_i r)(1+s_i s)t(z_i - Z_0) \end{aligned} \quad . \tag{1}$$

In the IE, the magnetic vector potential $\boldsymbol{A}$ can be interpolated by:

$$\boldsymbol{A} = \sum_{n=1}^{N}\left(\sum_{i=1}^{8} \boldsymbol{N}_i^n A_i^n\right), \tag{2}$$

where, N is the order of series expansion, $\boldsymbol{N}_i^n$ is the shape function of IE defined by,

$$\boldsymbol{N}_i^n = \frac{1}{t^n}\left(f_i(r,s)\nabla r + g_i(r,s)\nabla s\right) \ \ (i=1,2,3,4), \tag{3}$$

$$\boldsymbol{N}_i^n = \frac{n}{t^{n+1}} h_i(r,s)\nabla t \ \ (i=5,6,7,8). \tag{4}$$

The explicit form of $f_i(r,s)$, $g_i(r,s)$ and $h_i(r,s)$ are shown in Table I.

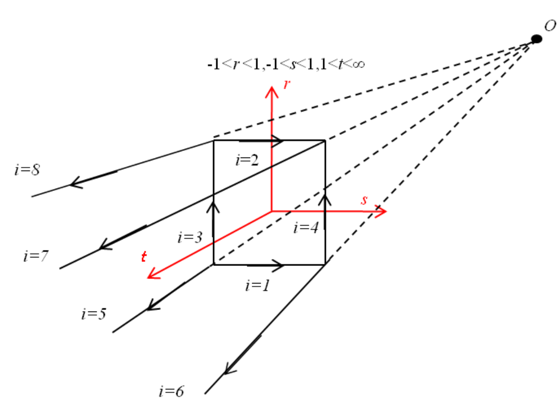

Fig.1. Local coordinates of infinite element

TABLE I EXPLICIT FORM OF (3) AND (4)

Edge number i	$f_i(r,s)$	$g_i(r,s)$	Edge number i	$h_i(r,s)$
1	$(1-s)/4$	0	5	$(1-r)(1-s)/4$
2	$(1+s)/4$	0	6	$(1+r)(1-s)/4$
3	0	$(1-r)/4$	7	$(1+r)(1+s)/4$
4	0	$(1+r)/4$	8	$(1-r)(1+s)/4$

III. DEFLATION TECHNIQUES

Let us consider solving an FE equation $K\boldsymbol{x} = \boldsymbol{b}$, where K is a symmetric semi-positive definate matrix whose eigenvalues are $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$, $\boldsymbol{b}$ is assumed to be in the range of K. The deflation technique is implemented the decomposition $\boldsymbol{x} = P\boldsymbol{x} + Q\boldsymbol{x}$, where P and Q are a projector matrix. P and Q satisfies $P^2 = P$ and $Q = I - P$. It is readily shown that PQ=0. Let us introduce the matrices W = [$\boldsymbol{w}_1, \cdots, \boldsymbol{w}_k$] and $\overline{W}$ = [$\boldsymbol{w}_{k+1}, \cdots, \boldsymbol{w}_n$] which are composed of the orthogonal eigenvectors of K. Then $\boldsymbol{x}$ is decomposed into the slowly and fast convergent components $W\boldsymbol{z}$, $\boldsymbol{x}$ - $W\boldsymbol{z}$, where the former is the projection of $\boldsymbol{x}$ onto the space spanned by $\boldsymbol{w}_i$. The vector $\boldsymbol{z}$ can be determined from the orthogonality condition $(\boldsymbol{x} - W\boldsymbol{z}, \boldsymbol{w}_i)_K$=0, that is

$$W^t K W \boldsymbol{z} = W^t K \boldsymbol{x}. \tag{5}$$

The above decomposition is expressed as $P\boldsymbol{x}$ = $\boldsymbol{x}$-$W\boldsymbol{z}$. thus, assuming W^tKW is regular, we obtain $P = I - W(W^tKW)^{-1}W^tK$. The equation for $P\boldsymbol{x}$ can be obtained from the commutability $KP = P^tK$, that is

$$P^t K\boldsymbol{x} = P^t \boldsymbol{b}. \tag{6}$$

It follows from $P^tKW = 0$ and $P^tK\overline{W} = K\overline{W}$ that P^tK has an effective condition number $\lambda_n / \lambda_{k+1}$ which is smaller than the condition number of K.

IV. NUMERICAL RESULTS

To shows the effects of the deflation technique, we analyzed a simple magnetostatic model which consists of a coil and air region shown in Fig.2. The number of IE, FE and unknowns are provided in Table II. The hexahedral edge FEs and the square IEs with N = 2 are used for mesh.

Figure 3 shows the eigenvalue distribution of matrix K except zero eigenvalues. We can see that there is a large difference between minimum eigenvalue and maximum one. Moreover, the distribution is clearly separated into two parts. The number of iterations in the deflated ICCG compared with ICCG is shown in Table III, and the convergence history is shown in Fig. 4. The convergence characteristic of deflated ICCG is significantly improved because the lower part of eigenvalues in Fig. 3 is deflated.

TABLE II MODEL PARAMETERS

	Finite element	Infinite element
Number of elements	1000	600
Number of unknowns	3630	3604

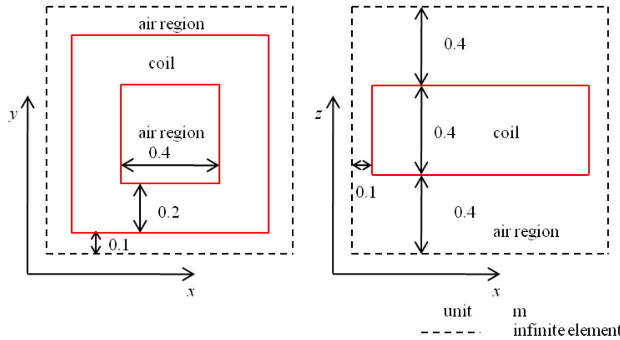

Fig.2. Analysis model

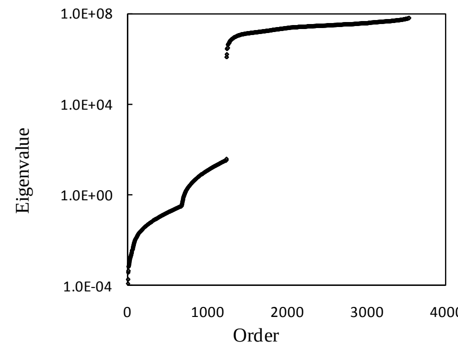

Fig.3. Eigenvalue distribution

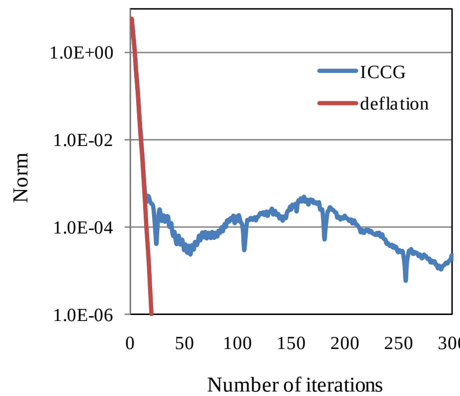

Fig.4. Comparison of convergence between ICCG and deflated ICCG

TABLE III NUMBER OF ITERATIONS

	ICCG	Deflated ICCG
Number of iterations	20	407

V. REFERENCES

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